Improving Performance of Multithreshold Decoder over Binary Erasure Channel

Grinchenko N., Gromov A., Ovechkin G.
Faculty of Computer Engineering
The Ryazan State Radio Engineering University
Ryazan, Russian Federation
ovechkin.g.v@rsreu.ru

Abstract—**The article discusses self-orthogonal error-correcting codes (SOC) for the decoding of which multithreshold algorithms (MTD) are usually used. The work gives the results of the research of MTD efficiency in binary erasure channels. Also new, longer and more efficient convolutional SOCs are received, comparative analysis of MTD efficiency and the efficiency of best methods to recover erasures is made. It is offered new concatenated code consisting of SOC and parity-check code for binary erasure channel allowing to decrease bit unrecovered erasure probability in the area of MTD efficient operation to 2..3 decimal orders. Lower bounds on bit unrecovered erasure probabilities for a decoder of the concatenated code are given; the its simulation results are discussed.**

Keywords—communication systems, self-orthogonal codes, multithreshold decoder, iterative decoding, parity-check codes, concatenated codes, binary erasure channels, erasure recovering

I. INTRODUCTION

One of the most important problems while developing high-throughput communication systems is a correct choice of methods to encoding and decoding error-correcting codes necessary to organize valid transmission of digital information. Nowadays the coding theory includes a lot of classes of error-correcting codes differing in many parameters. The survey of most prospective coding methods according to the criterion "efficiency-performance" has shown that one of the best ones for high-throughput channels is multithreshold decoder (MTD) for self-orthogonal codes (SOC) [1, 2, 3]. MTD being the modification of a threshold Massey decoder allows to decode even extremely long codes with linear from code length complexity. Dozens of publications about MTD view the efficiency of its application in binary symmetric or Gaussian channels. Especially interesting ones are also erasure channels the usage of which allows to significantly simplify the decoding process compared to the case of error-correcting. Such channels are suitable for MTD algorithm versions [3], which provide erasure recovering with the efficiency close to the efficiency of an optimal decoder and have minimum possible linear implementation complexity.

Besides MTD a variety of other codes which have efficient erasure recovering algorithms can be named. They include turbo codes [4], low-density parity-check (LDPC) codes [5], repeat-accumulate (RA) codes [5], accumulate-repeat-accumulate (ARA) codes [6], polar codes [7] and the family of fountain codes [8], being rateless erasure codes: tornado codes [9] becoming the basis for fountain codes, online codes [10], raptor codes [11], Luby transform (LT) codes [9]. They are used with iterative decoding algorithms meaning that in their limit they are capable to achieve channel capacity. But for some of these codes even several years later since their invention it is difficult for engineers and scientists to find comprehensive diagrams showing the dependency of bit unrecovered erasure probability by decoder from erasure probability in a channel. These diagrams give only limiting theoretical characteristics that complicate the evaluation of their application prospects in real devices that transmit and store data. The disadvantages of rateless erasure codes application are the inevitability to buffer relatively large data volume both in a transmitter and a receiver, the necessity to synchronize a sender and a receiver to generate random numbers or the transmission of additional information over the channel. It should be taken into consideration that these codes are low efficient in the channels where error symbol reception is found besides erasures. We should also note that using the codes of finite length and practically implemented decoding algorithms the characteristics achieved get a little worse.

A given work considers MTD for binary erasure channel, presents its efficiency estimation, performs the comparison of simulation results between MTD and other erasure recovery methods, offers concatenated code for erasure recovery based on SOC and parity-check code allowing to improve probability characteristics of MTD in such conditions.

The rest part of the paper is arranged as follows. In Section II we discuss the background of MTD. In section III we present simulation results for MTD over the binary erasure channel. The new concatenated code based on SOC is offered and investigated in section IV. Section V concludes this paper.

II. MILTITHRESHOLD DECODING SELF-ORTHOGONAL CODES FOR ERASURE CHANNELS

MTD allows decoding convolutional and block SOCs which are set using generator polynomials g_{ij}, determining the connections between *i*-th information and *j*-th check branches [3]. The example of a scheme of an encoder for a convolutional code with one information and one check branches set by polynomial $g_{11}=1+x+x^4+x^6$ is presented in Fig. 1. A given code is characterized by code rate R=1/2 and code distance d=5.

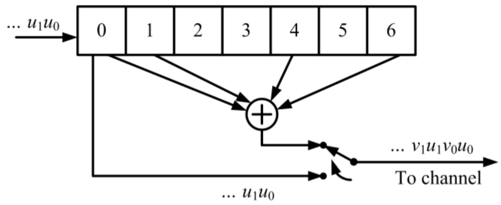

Figure 1. Scheme of convolutional SOC encoder

In a classic model of binary erasure channel (BEC) each bit can be transmitted correctly with the probability of $1–P_s$ or erased with the probability of P_s. The capacity of such channel equals $C=1–P_s$. MTD operation in erasure channel is as follows. First, the same as in the binary symmetric channel checks (syndrome bits) are calculated:

$$s_j = \sum_{p\in\Theta_j} u_p + v_j \mod 2 . \qquad (1)$$

Here u_p is a *p*-th element of information vector **u** received from the channel; v_j is *j*-th element of parity-check vector **v** received from the channel; Θ_j is a set of information bit numbers participating in the formation of *j*-th check bit. Here we should note that information and parity-check bits erased in a channel are not used during check calculation, their number is stored in a special register (erasure register **r**).

Later in the process of some erased information bit u_k decoding among all checks relating to it the one s_j containing only one erasure (r_j=1) is searched. It is evident that this erasure is caused by a decoded information bit u_k which can be easily recovered following the value of syndrome bit s_j with a single erasure in accordance with (1). At the same time it is necessary to perform the correction of all checks for a recovered information bit and to decrease by one the number of erasures in erasure register **r** for these checks. After that the decoding of the next bit starts. If there is no check containing only one erasure for erased bit then this bit is neglected and the transition to next erased information bit decoding takes place.

It should be noted that after the first recovering attempt it is possible that some erased information bits failed to be recovered. But according to algorithm meaning it doesn't enter any errors into the message. The successful attempt to recover one of the erasures for some symbol u_m leads to the decrease of the erasures left. So the attempt of repeated decoding of the message containing less number of erasures is justified.

As the correction of an information bit requires only one correctly received parity-check bit without any other erased information bits entering it MTD will operate in BEC in the conditions of higher probabilities of erasure compared to the channels where only errors take place. We should also note that MTD algorithm described is easily generalized in case when *q*-ary symbols (e.g., bytes) are used instead of bits.

To estimate the potential of MTD in BEC a lower bound on bit unrecovered erasure probability can be used together with optimal decoder (OD) for the SOC used [3]. OD in BEC must find a codeword which contains minimum number of erasures and coincides with all known bits of the message received. Lower bound on bit unrecovered erasure probability equals

$$P_{sOD} = P_s^d , \qquad (2)$$

where d is code distance of the SOC used. This bound is received only in the conditions when information bit will not be recovered if this bit and all checks relating to it are erased.

III. THE PERFORMANCE OF MULTITHRESHOLD DECODER OVER ERASURE CHANNEL

Fig. 2 presents the simulation results received after the research of the SOC code distance influence on the efficiency of MTD operation in BEC. This figure shows in solid lines the dependencies of bit erasure probabilities after MTD with 20 decoding iteration from erasure probabilities P_s in BEC for simple block codes having the length n=16000 bits, code rate R=1/2 and different values of code distance d (from 7 to 17).

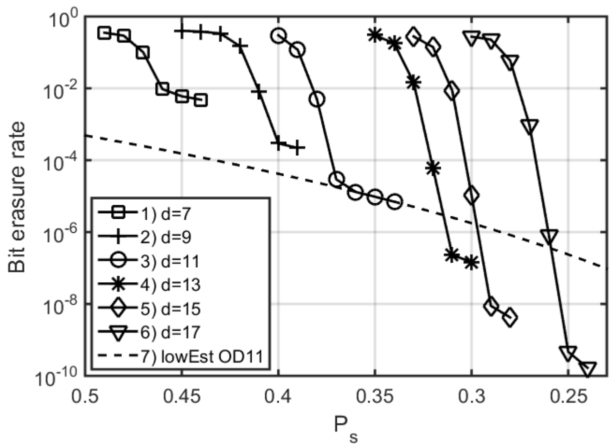

Figure 2. MTD characteristics in erasure channel for block SOC with different values of code distance d

Also Fig. 2 presents in dotted line *7* a bit unrecovered erasure probability by OD for the code with d=11 received in accordance with the expression (2) which shows that beginning with erasure probability equal to 0,37 MTD for the code with d=11 closely approaches the efficiency of an optimal decoder. The figure also shows that the increase of code distance d leads to the decrease of bit unrecovered erasure probability, but the area where MTD provides practically optimal decoding is shifted to the area of less erasure probability moving away from channel capacity being equal to 0,5 for a given code.

The results received after the research of code length n influence on the efficiency of decoding are presented in Fig. 3. The results were received using SOC with d=11, R=1/2. Given results show that during code length increase MTD is capable to provide the characteristics closer to the ones of optimal decoder in the conditions of higher erasure probability. In particular, during code length increase from 8000 to 40000 bits a threshold value of erasure probability when the decoding close to optimal is provided increases from 0,36 to 0,38.

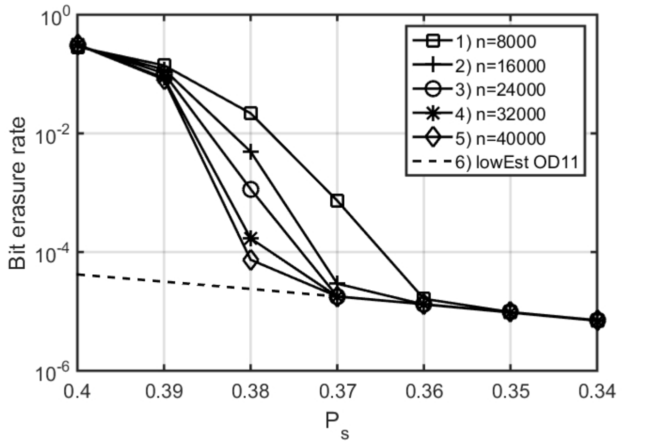

Figure 3. MTD characteristics in erasure channel for block SOC with different code length n

Further we shall consider the characteristics provided by the best known methods enumerated in introduction in BEC. These characteristics are presented in Fig. 4 where the dependencies of bit erasure rate after decoding from erasure probability P_s in BEC are shown. We should also note that the figure shows all codes with code rate equal to 1/2.

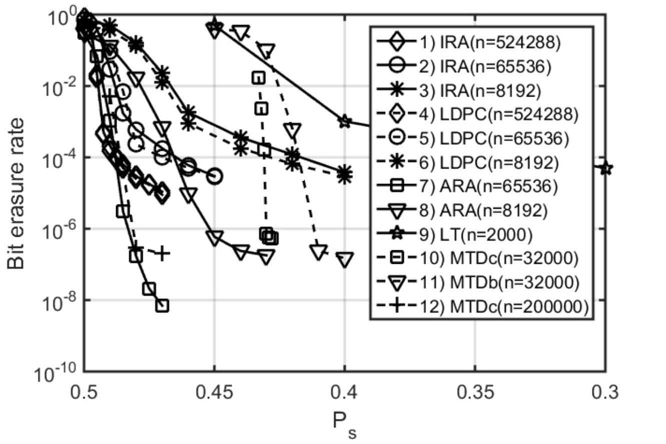

Figure 4. Characteristics of efficient erasure recovery methods for the codes with code rate 1/2

Curves *1*, *2* and *3* reflect the efficiency of non-regular RA codes with the length of 524288, 65536 and 8192 bits correspondingly while performing 50 decoding iterations; curves *4*, *5* and *6* reflect the efficiency of LDPC codes with the length of 524288, 65536 and 8192 bits correspondingly while performing 126 decoding iterations; curves *7* and *8* show the characteristics of ARA codes with the length of 65536 and 8192 bits used together with parity-check outer code; curve *9* shows the characteristics of LT code decoder with the length of 2000 bits. We should note that the usage of very long IRA and LDPC codes (n=524288) has allowed to receive bit unrecovered erasure probability equal to 10^{-4} having channel erasure probability 0,485 which is considered to be a serious result. But practical application of such long codes is quite problematic. Shorter codes with the length of 8192 bits provide the same bit unrecovered erasure probability in case when channel erasure probability equals 0,42..0,43. The curves *10* and *11* shows MTD characteristics for convolutional and block codes with the length of 32000 bits having code rate 1/2 at performing 20 decoding iterations. It should be noted that MTD for SOC are only slightly inferior to decoders for best known codes in their recovery capability. But its computational complexity turns out to be hundreds times less than the complexity of decoding algorithms of the codes considered. In addition it is important to mention that increasing the constraint length of the convolutional code used up to 200000 bits, optimizing its structure and increasing the number of decoding iterations up to 90 we improve MTD characteristics substantially which is shown by curve *12* in Fig. 4. A given MTD provides the efficiency comparable to best known codes and has much less computing complexity.

IV. The Concatenation of Self-Orthogonal Codes with Parity Check Codes

The results received after the research of MTD efficiency in BEC have shown that they allow to provide the decoding with the efficiency being close to optimal one. But the area of erasure probabilities where MTD operates nearly as OD depends on the code distance of the code used: the less is the code distance, the higher is the erasure probability for MTD to operate. But the codes with small code distance are not capable to provide low bit unrecovered erasure probability which one is required in high reliable communication systems.

It is doubtless that one of the most efficient approaches to increase the validity of data transmission is the application of concatenated codes. To solve this problem the work offers to use concatenated scheme where together with internal SOC having code rate R_2 outer parity-check code (PCC) with code rate R_1 is used. R_1 must be as close to 1 as possible to leave the redundancy at the same level. Structural scheme of data transmission system with concatenated code offered is presented in Fig. 5. It should be noted that generally iterative work of MTD and PCC decoder where they interchange their decisions can be organized.

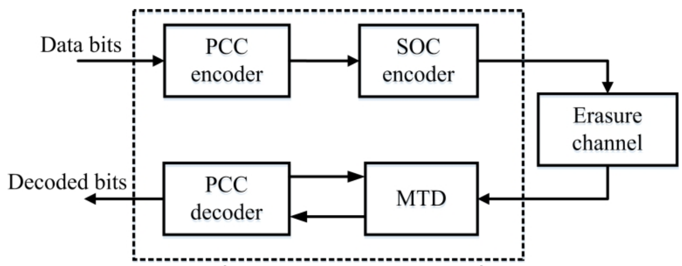

Figure 5. Simplified structural scheme of data transmission system with the concatenated code offered

To estimate the efficiency of concatenated code offered it is useful to receive the lower bound on bit unrecovered erasure probability after PCC decoder. Usually PCC are used only for single error detection but in BEC it is capable to correct single erasure. This becomes possible when the sequence with only one symbol erased enters PCC decoder input. So bit unrecovered erasure probability is estimated

$$P_{ePCC} = \sum_{i=2}^{n} \frac{i}{n} C_n^i p^i q^{n-i}, \qquad (3)$$

where $C_n^i = \dfrac{n!}{i!(n-i)!}$; p is erasure probability in PCC decoder input; q=1–p is the probability to receive non-erased bit by PCC decoder. It should be noted that bits enter PCC decoder after they leave MTD. As it was shown above, MTD is capable to provide the decoding being close to optimal. That's why bit erasure probability in PCC decoder input can be estimated using the expression (2). As a result lower bound on bit unrecovered erasure probability for the concatenated code offered will be determined by the

$$P_{eC} = \sum_{i=2}^{n} \frac{i}{n} C_n^i (P_s^d)^i (1-(P_s^d))^{n-i}. \qquad (4)$$

Using the expression (4) lower bounds on bit unrecovered erasure probabilities after decoder of the concatenated code including SOC with code distance d=11 and PCC have been received. The Fig. 6 shows the curves *2, 3, 4* received for PCC usage with the length of 25, 50 and 100 bits correspondingly, and the curve *1* corresponds to lower bound of unrecovered erasure probability of an optimal decoder for the SOC. The figure shows that the usage of the concatenated code leads to the decrease of bit unrecovered erasure probability of three or more orders of magnitude compared with SOC. The less PCC length the decoder characteristics are better.

The simulation results for SOC and concatenated code offered using designed SOC with c d=11, R=1/2, n=36000 bits and PCC length equal 50 bits are presented in Fig. 6 by the curves *5* and *6* correspondingly. PCC decoder was used after the last MTD iteration. The comparison of lower bounds and simulation results shows that these bounds are suitable for preliminary estimation of the concatenated code efficiency. The comparison of the curves *5* and *6* also shows that the concatenated code beginning with erasure probability equal to 0,41, provides 3 orders less bit unrecovered erasure rate in comparison with SOC. It should be noted that PCC decoder usage after each MTD iteration can slightly approximate the area of efficient decoder operation to channel capacity that is illustrated by the curve *7* in Fig. 6. Using PCC with more efficient SOC, e.g., the characteristics of which were shown in Fig.4 by the curve *12* bit unrecovered erasure probability of 10^{-12} order at erasure probability 0,48 can be provided. In this case the characteristics will be a little better than the characteristics of the codes known.

It should be noted that in case when PCC length is 50 bits, the total code rate is decrease to 2% in comparison with SOC. The increase in the number of elementary operations per bit while decoding concatenated code offered in comparison with MTD is less than 1%.

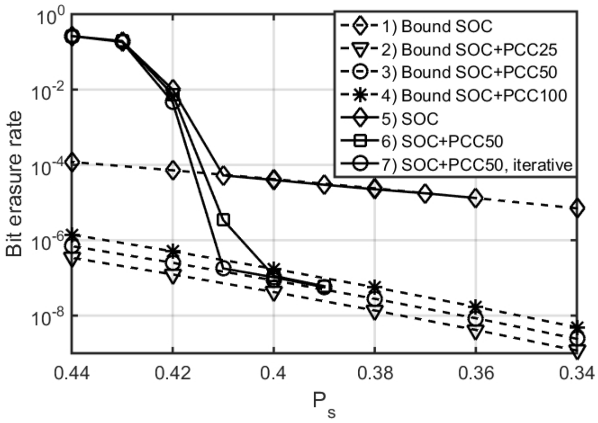

Figure 6. The characteristics of the decoder for concatenated code offered

V. Conclusion

The work makes the research of MTD in BEC at different code parameters (code type, length, code distance). It is shown that MTD efficiently recovers erasures operating close to erasure channel capacity and decreases bit unrecovered erasure probability on several orders using concatenated code offered and preserving low implementation complexity. This fact makes it highly competitive with best known erasure recovery methods used in high-throughput communication systems.

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