The Performance of Concatenated Schemes Based on Non-binary Multithreshold Decoders

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Abstract. Symbolic *q*-ary multithreshold decoding (*q*MTD) for *q*-ary self-orthogonal codes (*q*SOC) is analyzed. The SER performance of *q*MTD is shown to be close to the results provided by optimum total search methods, which are not realizable for non-binary codes in general. *q*MTD decoders are compared with different decoders for Reed-Solomon and LDPC codes. The results of concatenation of *q*SOC with simple to decode outer codes are described. The complexity of *q*MTD is also discussed.

Keywords: iterative decoding, non-binary (symbolic) multithreshold decoding, *q*-ary self-orthogonal codes, concatenated codes, symbolic codes

1 Introduction

Error correcting coding is used to correct errors appearing during data transmission via channels with noises. Main attention in literature is given to binary error-correcting codes working with data on the level of separate bits. But in many digital systems it's often more convenient to work with byte structure data. As an example, it's more convenient to work with bytes in systems which store big volumes of data (optic discs and other devices). In such systems to protect data from errors it is recommended to use non-binary error-correcting codes. At present preference among non-binary codes is given to Reed-Solomod codes (RS), which have algebraic decoding algorithms [1], allowing to correct up to half-distance errors as well as more complex algorithms [2] providing correction of higher error number. At the same time due to their implementation complexity such methods allow to decode only short and thus low-effective RS codes. Lately many specialists have been developing decoders of non-binary low-density parity-check (*q*LDPC) codes which are able to provide extremely high efficiency [3, 4]. But the complexity of such decoders especially with big alphabet size still remains too high to be used in practice.

Special attention among non-binary algorithms to correct errors should be given to non-binary (q-ary) self-orthogonal codes and special high-speed alphabet multithreshold decoders (qMTD) corresponding to them [5…8], being the development of binary multithreshold decoders (MTD) [5, 6, 9…10]. Great interest to MTD is shown not only in Russia [11, 12]. Research results given in [5…8] show that qMTD greatly exceed in their efficiency RS codes and qLDPC codes being used in practice remaining as simple to be implemented as their prototypes – binary MTD**.** It is also very important not to use multiplication in non-binary fields during encoding and decoding as well as total independence of alphabet codes lengths from the size of applied symbols. That's the reason why such codes will find broad application in the sphere of processing, storage and transmission of large volumes of audio, video and other types of data.

The rest part of the article is organized in the following way. Section 2 contains basic information about qMTD. Section 3 shows the results of qMTD efficiency comparison with efficiency of decoders for RS and qLDPC codes. Section 4 is dedicated to the development of new concatenated schemes to correct errors based on qMTD and their efficiency analysis. Section 5 demonstrates the main conclusions.

2 Non-binary multithreshold decoding

Let's describe operating principles of qMTD during non-binary self-orthogonal codes (qSOC) decoding. The description is given for q-ary symmetric channel (qSC) having alphabet size q, $q > 2$, and symbol error probability p_0.

Let's assume linear non-binary systematic convolutional or block self-orthogonal code with parity-check matrix **H** to be equal to binary case [6, 13], i.e. it has only zeros and ones excluding the fact that instead of 1 there will be –1 in identity submatrix, i.e. $\mathbf{H} = [\mathbf{P} : -\mathbf{I}]$. Here **P –** submatrix defined by generator polynomial for binary SOC; **I** – identity submatrix. Generator matrix of such code will be of $\mathbf{G} = [\mathbf{I} : \mathbf{P}^{\mathrm{T}}]$ type. This code can be used with any size q of alphabet.

Note that for this qSOC during encoding and decoding operations only addition and subtraction on q module are necessary to be made. Calculations in non-binary fields are not applied in this case.

The example of a scheme realizing the operation of encoding by block qSOC, given by generator polynomial $g(x) = 1+x+x^4+x^6$, is given on Fig. 1. Such code is characterised by the parameters: code length n=26 symbols, data part length k=13 symbols, code rate R=1/2, code distance d=5.

Let's assume that encoder has performed encoding of data vector $\boldsymbol{U}$ and received code vector $\boldsymbol{A} = [\boldsymbol{U}, \boldsymbol{V}]$, where $\boldsymbol{V} = \boldsymbol{U} \cdot \mathbf{G}$. Note that in this example and below when multiplication, addition, subtraction of vectors and matrices are made, module arithmetics is applied. When code vector $\boldsymbol{A}$ having the length n with k data symbols on qSC is transmitted decoder is entered with vector $\boldsymbol{Q}$, generally speaking, having differences from original code vector due to errors in the channel: $\boldsymbol{Q} = \boldsymbol{A} + \boldsymbol{E}$, where $\boldsymbol{E}$ – channel error vector of qSC type.

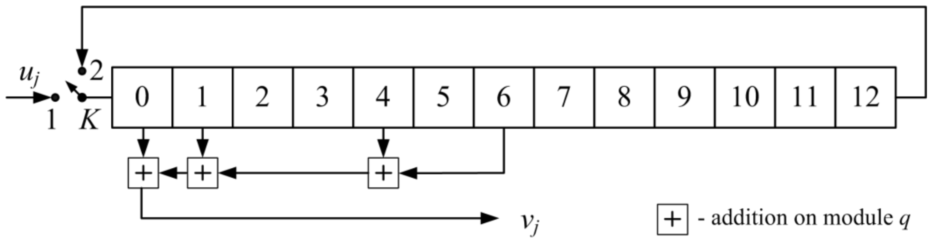

Fig. 1. Encoder for block qSOC, given by polynomial $g(x) = 1+x+x^4+x^6$

Operating algorithm of qMTD during vector $\boldsymbol{Q}$ decoding is the following [5…8].

1. Syndrome vector is calculated $\boldsymbol{S} = \mathbf{H} \cdot \boldsymbol{Q}^{\mathrm{T}}$. Difference register $\boldsymbol{D}$ is reset. This register will contain data symbols changed by decoder. Note that the number of non-zero elements of $\boldsymbol{D}$ and $\boldsymbol{S}$ vectors will always determine the distance between message $\boldsymbol{Q}$ received from the channel and code word being the current solution of qMTD. The task of decoder is to find such code word which demands minimal number of non-zero elements of $\boldsymbol{D}$ and $\boldsymbol{S}$ vectors. This step totally corresponds to binary case.

2. For arbitrarily chosen decoded q-ary data symbol i_j of the received message let's count the number of two most frequent values of checks s_j of syndrome vector $\boldsymbol{S}$ from total number of all checks relating to symbol i_j, and symbol d_j of $\boldsymbol{D}$ vector, corresponding to i_j symbol. Let the values of these two checks be equal to h_0 and h_1, and their number be equal to m_0 and m_1 correspondingly when $m_0 \geq m_1$. This step is an analogue of sum reception procedure on a threshold element in binary MTD.

3. If $m_0 - m_1 > T$, where T – a value of a threshold (some integer number), then from i_j, d_j and all checks regarding i_j error estimation equal to h_0 is subtracted. This step is analogous to comparison of a sum with a threshold in binary decoder and change of decoded symbol and correction via feedback of all syndrome symbols being the checks for decoded symbol.

4. The choice of new i_m, $m \neq j$ is made, next step is clause 2.

Such attempts of decoding according to cl. 2…4 can be repeated for each symbol of received message several times [5, 6]. Note that when implementing qMTD algorithm the same as in binary case it is convenient to change all data symbols consequently and to stop decoding procedure after fixed number of error correction attempts (iterations) or if during such iteration no symbol changed its value. The example of qMTD implementation for encoder from Fig. 1 is given on Fig. 2.

For qMTD algorithm described the following theorem is valid,

Theorem. Let decoder realize qMTD algorithm for the code described above. Then during each change of decoded symbols a transition to a more plausible solution in comparison with previous decoder solutions takes place.

Proof of a theorem is given in [5…8]. In the course of proof it is shown that total Hamming weight of syndrome and difference registers during each change of decoded symbols in accordance with qMTD algorithm described above strictly decreases.

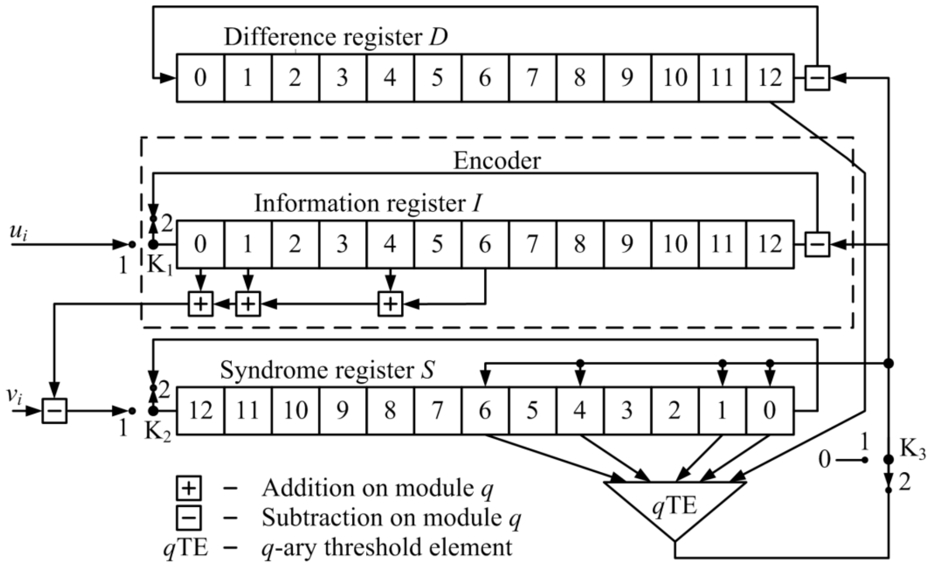

Fig. 2. MTD for block *q*SOC.

Let's note two most important features characterizing offered algorithm. First, as in case of binary codes we can't claim that *q*MTD solution improvement during multiple decoding attempts will take place till optimal decoder solution is achieved. In fact both in block and in convolutional codes it's possible to meet such error configurations which cannot be corrected in *q*MTD, but some of them can be corrected in optimal decoder. That's why the main way to increase *q*MTD efficiency is to search codes where these noncorrected error configurations are quite rare even in high level of noise. The questions to choose such codes are considered in detail in [6].

Another important moment is the fact that in comparison with traditional approach to major systems to change decoded symbol *q*MTD needs not absolute but relatively strict majority of checks as it follows from $m_0 – m_1 > T$ condition. E.g., in *q*SOC with $d = 9$ an error in decoded symbol will be corrected even when only 2 checks will be correct from 9 his checks (including symbol d_j of difference register) and the other 7 - erroneous! This situation cannot be imagined for binary codes but for *q*MTD this is typical.

These features essentially expand the possibilities of non-binary multithreshold algorithm during operation in high noises retaining as it follows from given description only linear dependence of implementation complexity from code length.

3 Simulation results

Let's compare characteristics of *q*MTD and other non-binary error correction methods in *q*SC. Dependencies of symbol error rate P_s after decoding from symbol error P_0 probability in *q*SC for codes with code rate R=1/2 are given in Fig. 3. Here curves *5* and *6* show characteristics of *q*MTD for *q*SOC with block length n=4000 and

32000 symbols when using 8-bit symbols (alphabet size q=256). The volume of simulation in lower points of these graphs contained from $5 \cdot 10^{10}$ to $2 \cdot 10^{12}$ symbols which shows extreme method simplicity. As a comparison in this Figure curve *1* shows characteristics of algebraic decoder for (255, 128) RS code for q=256. As it follows from the Fig. 3, efficiency of qMTD for qSOC turns out to be far better than efficiency of RS code decoders using the symbols of similar size. When code length in qMTD increases the difference in efficiency turns out to be even higher. Note that even when using concatenated schemes of error correction based on RS codes it's not possible to increase decoding efficiency considerably. E.g., with the help of product-code having code rate 1/2, consisting of two RS codes with q=256 and several dozens of decoding iterations error rate less than 10^{-5} can be provided with error probability in the channel only equal to 0,18 (curve *4* in Fig. 3), which is considerably worse than when using qMTD. Besides different methods to increase correcting capability of RS codes including all variations of Sudan algorithm ideally have the complexity of n^2 order. For the codes having the length of 32000 symbols this leads to the difference in complexity equal to 32000 times having at the same time little increase of error-correctness. This is shown in Fig. 3 with curve *3*, which gives the estimations of Wu [2] algorithm possibilities for (255, 128) of RS code.

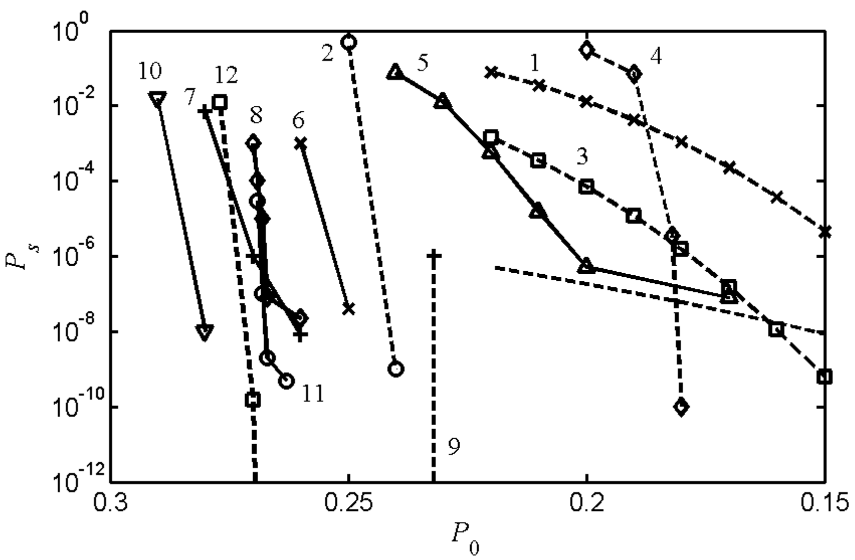

Fig. 3. Characteristics of non-binary codes with code rate R=1/2 in qSC

Additional advantage of qMTD over other error correcting methods is the fact that it allows to work easily with symbols of any size providing high correcting capability. This is confirmed by curves which show characteristics of qMTD for code having the length equal to 32000 two-byte symbols (curve *7*) and to 100000 four-byte symbols (curve *10*). We should note that very simple to be implemented qMTD decoder for two-byte code with the length 32000 is capable to provide error-correctness not accessible even by RS code with the length of 65535 two-byte symbols (curve *2* in Fig. 3), the decoder for which is not to be implemented in close future. Besides, qMTD for four-byte symbols even surpasses in efficiency more complicated decoder of qLDPC codes with the length of 100000 four-byte symbols which has the example of characteristics presented in Fig. 3 by curve *9* [4].

It should be noted once more that to achieve these results with the help of qMTD used codes should be chosen very thoroughly and the main criterion while choosing should be the degree of resistance to the effect of error propagation. At the same time the most effective are the codes where several data and several check branches are used [6, 14]. In [15] it is shown that in the process of such codes optimizing it is possible to improve qMTD operating efficiency. Particularly, characteristics of the code with q=256 and code rate 1/2 found in [15] are given in Fig. 3 by curve *8*. It is clear that this code provides effective work in conditions of bigger error probabilities in qCK, than the codes known before (curve *6*), having the same complexity of their decoding.

4 Concatenated schemes of error correction based on *q*MTD

One of the ways to improve qMTD characteristics is to use it in concatenated encoding schemes. The simplest and most effective concatenated encoding scheme is concatenated scheme on the basis of qSOC and control code on module q [6, 8, 16]. In the field of its effective work qMTD is known to leave only rare single errors. The task to correct such single errors is easily solved with the help of control codes on module q.

The process of encoding by concatenated code encoder on the basis of qSOC and control code on module q is the following. First each sequence consisting of n–1 symbols is complemented by such n-th symbol that the sum of symbols value having the sequence of n elements on q module becomes equal to 0. After that this new sequence of n elements is encoded by qSOC encoder. Decoding process of the message received from the channel is made in reverse order, i.e. non-binary multithreshold decoding is made first after which in the conditions of lower noise level channel contains basically single errors which are corrected by decoder for control code on module q.

Operation of decoder for control code on module q is the following. First e sum on module q values for block consisting of n elements is calculated. If this sum is not equal to 0, then among the first n–1 elements in the block the one with less reliability should be found, the reliability of which is less than reliability of n-th symbol in block. If such symbol exists then it is changed on e value. The reliability here is understood as m_0–m_e difference, where m_0 – number of zero symbols of syndrome and difference register of qMTD connected with given data symbol; m_e – number of symbols of syndrome and difference register of qMTD with the value e and connected with given symbol.

In Fig. 3 curve *11* shows characteristics of concatenated encoding scheme consisting of qSOC and control code on module q in qSC. Inner code was qSOC with minimum code distance d=17 and code rate R=8/16 the characteristics of which are represented by curve *8*. Outer code was control code on module q with the length L=50. During qSOC decoding qMTD with 30 iterations was used. The Figure shows that usage of decoder for control code on module q with block length L=50 after qMTD allows to reduce decoding error rate on more than two orders. The increase of calculation volume in concatenated code is less than 20% in comparison with original qMTD algorithm.

Essential drawback of the concatenated scheme described above is the fact that decoder of outer control code on module q sometimes does not correct even the only error in the block. To eliminate this drawback it is recommended to use together with qSOC more effective and simple to be implemented non-binary code the decoder of which will always correct the only symbol error in the block. This will allow to reduce error rate in the field of effective qMTD operation even more in comparison with concatenated scheme presented above. As an example of such code non-binary Hamming codes [17] can be used. At the same time known non-binary Hamming codes have such features as the necessity to use extended Galois fields in the process of decoding as well as dependence of code length from alphabet size. As a result the application of such codes in offered concatenated scheme especially when the alphabet is big becomes too complicated. That's why it could be offered to build non-binary Hamming codes [16] on the basis of known binary Hamming codes. Let's describe them in detail.

Parity-check matrix of these codes coincides with parity-check matrix of binary Hamming codes $\mathbf{H}_h=[\mathbf{C}_h : \mathbf{I}]$. Generator matrix will be as follows $\mathbf{G}_h=[\mathbf{I} : -\mathbf{C}_h^T]$. Let us formulate the principles to decode this code.

Let's assume that after qMTD vector $\boldsymbol{Y}$ entered input of non-binary Hamming code decoder. In the process of decoding a syndrome of received message is calculated first:

$$\boldsymbol{S}_h = \boldsymbol{Y} \cdot \mathbf{H}_h^T.$$

If the received message contains only one error with value e_j on j position then generated syndrome can be written down as

$$\boldsymbol{S}_h = \boldsymbol{S}_2^j e_j,$$

where $\boldsymbol{S}_2^j$ – syndrome of binary Hamming code with single error on j position. Consequently, such symbol of received message need to be corrected on value e_j for which a column of parity-check matrix $\mathbf{H}_h$ coincides with syndrom $\boldsymbol{S}_2^j$.

If received message contains two errors e_i and e_j on i and j positions then the syndrom can be written down as follows

$$\boldsymbol{S}_h = \boldsymbol{S}_2^i e_i + \boldsymbol{S}_2^j e_j.$$

Such syndrome contains only values 0, e_i, e_j and e_i+e_j. Consequently, such symbol of received message need to be corrected on value e_i for which matrix column $\mathbf{H}_h$ coincides with vector $\boldsymbol{S}_2^i$, and such symbol of received message need to be corrected on value e_j for which matrix column $\mathbf{H}_h$ coincides with vector $\boldsymbol{S}_2^j$. Thus, offered algorithm of offered non-binary Hamming codes decoding in majority of cases (approximately in 71% cases for q=256 [16]) is able to correct even two errors. And if it is use offered extended non-binary Hamming codes having in addition one general check on module q then two errors are practically corrected in all cases (in 99% of cases for q=256 [16]).

The example of performance of offered concatenated scheme containing qSOC with R=8/16, q=256, d=17 and given extended non-binary Hamming code with the length N_2=128 is shown in Fig. 3 by curve *12*. At the same time total decoding complexity due to addition of extended non-binary Hamming code increases not more than 35 % [16]).

5 Conclusion

Given results allow to conclude that qMTD methods can really be regarded as unique algorithms capable to provide effective decoding in the conditions of high noise level requaring quite small number of operations and achieving highest levels of reliability in the process of digital information transmission and storage as well as its processing rate in high-speed communication channels and in the devices to store large volume of data.

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